Lecture 11
14.2/14.3 Limits, partial derivatives

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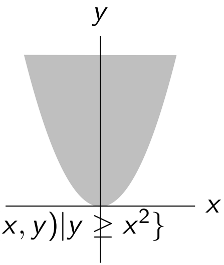
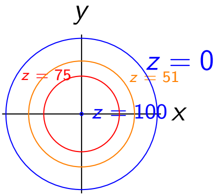
Things to note

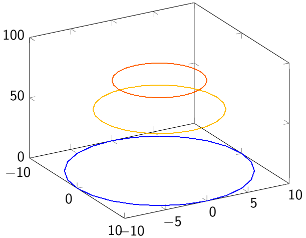
Exams solutions are posted.

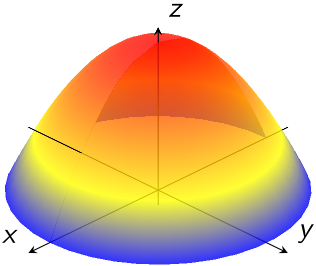
Last class



$$D = \{(x, y) | y \geq x^2\}$$







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Question

How many ways can you approach the number a along the number line?

Answer

Two ways: Either from the right (with larger numbers) or from the left (with smaller numbers).

Recall

Theorem

If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, *then* $\lim_{x\to a} f(x)$ *exists and is equal to the value of the limits in the equation.*

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$$\lim_{(x,y)\to(a,b)} f(x, y)$$

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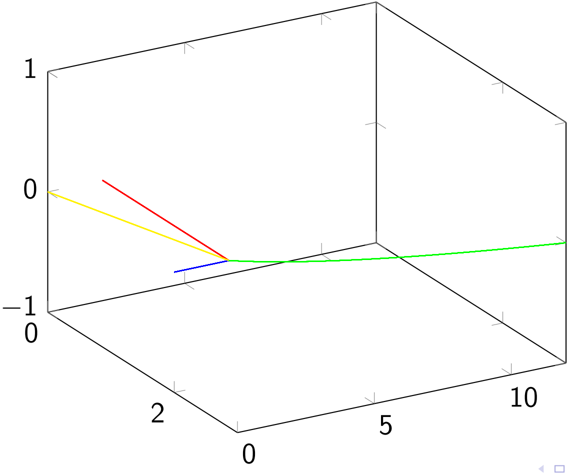

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Example

Find $\lim_{(x,y)\to(2,4)} \frac{x^2+y^2}{y-x}$.

We can simply "plug in" the values that x and y are heading to in the limit.

$$\lim_{(x,y)\to(2,4)} \frac{x^2+y^2}{y-x} = \frac{2^2+4^2}{4-2} = 10.$$

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$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1} = \lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{x(y-2)-(y-2)}{x-1}$$

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Use proper limit notation!

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$f(x,y)$ approaches the height L no matter what path approaching (a,b) in the domain is chosen.

This makes it manageable to show a limit doesn't exist: Just pick paths that give different values in the limit.

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Example

Let $f(x) = \dfrac{x^4 - y^2}{x^4 + y^2}$. *Show* $\lim_{(x,y)\to(0,0)} f(x,y)$ *doesn't exist.*

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First we pick the path $x = 0$. This is a valid path since the line $x = 0$ goes through $(0,0)$.

$$\lim_{(x,y)\to(a,b)} \frac{x^4 - y^2}{x^4 + y^2} \stackrel{\text{along } x=0}{=} \lim_{(0,y)\to(0,0)} \frac{0 - y^2}{0 + y^2} = -1.$$

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Thus the limit doesn't exist.